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N-Impurity superstring spectra near the pp-wave limit

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Abstract

The complicated non-linear sigma model that characterizes the first finite-radius curvature correction to the pp-wave limit of IIB superstring theory on $AdS_5 \times S^5$ has been shown to generate energy spectra that perfectly match, to two loops in the modified 't Hooft parameter λ', finite R-charge corrections to anomalous dimension spectra of large-R $\mathcal{N} = 4$ super Yang-Mills theory in the planar limit. This test of the AdS/CFT correspondence has been carried out for the specific cases of two and three string excitations, which are dual to gauge theory R-charge impurities. We generalize this analysis on the string side by directly computing string energy eigenvalues in certain protected sectors of the theory for an arbitrary number of worldsheet excitations with arbitrary mode-number assignments. While our results match all existing gauge theory predictions to two-loop order in λ', we again observe a mismatch at three loops between string and gauge theory. We find remarkable agreement to *all* loops in λ', however, with the near pp-wave limit of a recently proposed Bethe ansatz for the quantized string Hamiltonian in the $\mathfrak{su}(2)$ sector. Based on earlier two- and three-impurity results, we also infer the full multiplet decomposition of the N-impurity superstring theory with distinct mode excitations to two loops in λ'.

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1 Introduction

The AdS/CFT correspondence is currently the subject of a new generation of tests which endeavor to compare, near certain simplifying limits, the perturbative anomalous dimension spectrum of planar (large-N_c) $\mathcal{N} = 4$ $SU(N_c)$ super Yang-Mills (SYM) theory with the energy spectrum of non-interacting IIB superstring theory on $AdS_5 \times S^5$. This line of investigation was sparked by the discovery that a certain large R-charge limit of the gauge theory, the so-called BMN limit, corresponds to a Penrose limit of the string background geometry in which the metric of $AdS_5 \times S^5$ is reduced to that of a pp-wave [1], a background in which the string theory is free in lightcone gauge [2, 3]. This matching becomes much more elaborate when higher-order perturbative corrections in the 't Hooft coupling $\lambda = g^2_{\rm YM} N_c$ are included in the gauge theory [4, 5, 6, 7, 8], or when spacetime curvature corrections away from the pp-wave limit are admitted in the string theory [9, 10, 11, 12, 13]. Since the Penrose limit of the string theory is realized by boosting string states along an equatorial geodesic in the S^5 subspace, the latter corrections to the pp-wave geometry emerge in inverse powers of the S^5 string angular momentum J.

A number of important discoveries have advanced the gauge theory side of these studies. It has been realized, for example, that certain sectors of the $\mathcal{N} = 4$ dilatation generator can be mapped to one-dimensional integrable spin-chain Hamiltonians [14]. These sectors are typically labelled by the subalgebra of the full $\mathfrak{psu}(2,2|4)$ superconformal algebra under which they are invariant, and they can be modelled by spin-chain systems which are invariant under the same symmetry. The problem of computing operator anomalous dimensions in a given sector of the gauge theory can therefore be replaced with that of finding the energy eigenvalue spectrum of a corresponding spin chain. The integrability of these spin-chain systems implies that their energy spectra can be extracted via techniques such as the Bethe ansatz. This powerful tool was first applied in this context at one-loop order ($O(\lambda)$) in an $\mathfrak{so}(6)$ sector by Minahan and Zarembo [14], and generalized to the full superconformal symmetry algebra in [15]. (The Bethe ansatz approach to integrable systems is described in a more general setting in [16], for example.) The Bethe equations are usually exactly soluble for spin chains with two impurities (the spin-chain impurity number refers to the number of R-charge defects in the corresponding SYM operator, and is equivalent to the number of worldsheet excitations in the dual string theory). For higher impurity number, the one-loop Bethe equations can be solved perturbatively near the limit of infinite chain length, a limit which corresponds to the large-J, pp-wave limit of the string theory. Furthermore, since integrability has been shown to persist in the gauge theory to at least three-loop order (in an $\mathfrak{su}(2)$ sector), "long-range" Bethe equations have emerged encoding the higher-loop dynamics of the theory [7, 8]. By incorporating non-nearest-neighbor interactions of pseudoparticle excitations on the spin lattice that are typical of higher-loop SYM spin-chain Hamiltonians, such equations are in fact meant to describe the gauge theory physics to *all orders* in λ (the Inozemtsev chain of [7] failed to exhibit proper BMN scaling at four loops, but this shortcoming was circumvented by the modified ansatz of [8]). Predictions from the long-range $\mathfrak{su}(2)$ Bethe equations have recently been tested against a separate virial technique for

operators with three R-charge impurities in [17], where agreement was obtained to three-loop order. (The $\mathfrak{su}(2)$ spin-chain Hamiltonian has not been fixed definitively beyond this order.)

On the string theory side the first $1/J$ correction to the free pp-wave spectrum was computed in [9, 10, 11]. The string energies in this setting correspond in the gauge theory to the difference between operator scaling dimensions and R-charge ($\Delta \equiv D - R$), and states are arranged into superconformal multiplets according to the $\mathfrak{psu}(2,2|4)$ symmetry of the theory. The fully supersymmetric two-excitation (or two-impurity) system, for example, is characterized by a 256-dimensional supermultiplet of states built on a scalar primary. The complete spectrum of this system was successfully matched to corresponding SYM operator dimensions in [10, 11] to two loops in the modified 't Hooft coupling $\lambda' = \lambda/J^2$. A three-loop mismatch between the gauge and string theory results discovered therein comprises a long-standing and open problem in these studies, one which has appeared in several different contexts (see, eg., [7, 8, 18]). This was extended to the three-impurity, 4,096-dimensional supermultiplet of string states in [12], where precise agreement with the corresponding gauge theory was again found to two-loop order, and a general disagreement reappeared at three loops. In the latter study, three-impurity string predictions were compared with corresponding gauge theory results derived both from the virial technique described in [17] and the long-range Bethe ansatz of [15] (which overlaps at one loop with the original $\mathfrak{so}(6)$ system studied in [14]).

In the present study we generalize the string side of these investigations by computing, directly from the Hamiltonian, various N-impurity spectra of IIB superstring theory at $O(J^{-1})$ in the large-J curvature expansion near the pp-wave limit of $AdS_5 \times S^5$. We focus on the bosonic $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ sectors which are characterized by N symmetric-traceless bosonic string excitations in the S^5 and AdS_5 subspaces, respectively. These sectors are known to decouple from the theory to all orders in λ', and are thus referred to as "closed" sectors of the theory. Based on calculations in the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ sectors, we also formulate a conjecture for the N-impurity spectrum of states in a protected $\mathfrak{su}(1|1)$ sector composed of N fermionic excitations symmetrized in their $SO(4) \times SO(4)$ spinor indices. We then describe the complete supermultiplet decomposition of the N-impurity spectrum to two loops in λ' using a simple generalization of the two- and three-impurity cases.

We note that a new Bethe ansatz for the string theory was recently proposed by Arutyunov, Frolov and Staudacher [19] which is meant to diagonalize the fully quantized string sigma model in the $\mathfrak{su}(2)$ sector to all orders in $1/J$ and λ'. This ansatz was shown in [19] to reproduce the two- and three-impurity spectra of quantized string states near the pp-wave limit detailed in [11, 12]. The methods developed here allow us to check their formulas directly against the string theory for any impurity number at $O(J^{-1})$, and we find that our general $\mathfrak{su}(2)$ string eigenvalues agree to all orders in λ' with their $\mathfrak{su}(2)$ string Bethe ansatz!

Section 2 is dedicated to a brief review of the string system near the pp-wave limit. We compute the N-impurity energy spectra of the $\mathfrak{su}(2)$, $\mathfrak{sl}(2)$ and $\mathfrak{su}(1|1)$ closed sectors of this system in section 3, and generalize the complete N-impurity supermultiplet structure of the theory to two-loop order in λ' in section 4. We conclude in section 5 with a discussion of future problems.

2 Notation and string quantization in $AdS_5 \times S^5$

For convenience we will present the first curvature correction to the bosonic sector of the string Hamiltonian near the pp-wave limit of $AdS_5 \times S^5$, briefly review its derivation and introduce some standard notation. The spacetime metric of $AdS_5 \times S^5$ can be written as

$$ds^2 = R^2 \left[-\left(\frac{1+\frac{1}{4}z^2}{1-\frac{1}{4}z^2}\right)^2 dt^2 + \left(\frac{1-\frac{1}{4}y^2}{1+\frac{1}{4}y^2}\right)^2 d\phi^2 + \frac{dz_k dz_k}{(1-\frac{1}{4}z^2)^2} + \frac{dy_{k'} dy_{k'}}{(1+\frac{1}{4}y^2)^2} \right] , \qquad (2.1)$$

where R is the spacetime radius and the coordinates z_k $(k = 1, \ldots, 4)$ and $y_{k'}$ $(k' = 5, \ldots, 8)$ parameterize two transverse $SO(4)$ spaces which descend from the AdS_5 and S^5 subspaces, respectively. Lightcone coordinates are introduced by the reparameterization

$$t = x^+ \qquad \phi = x^+ + x^-/R^2 , \qquad (2.2)$$

with corresponding momenta

$$-p_+ = \Delta - J \qquad -p_- = i\partial_{x^-} = \frac{i}{R^2}\partial_\phi = -\frac{J}{R^2} . \qquad (2.3)$$

The S^5 angular momentum J is related to the scale factor R by $p_- R^2 = J$. To reach the pp-wave limit, the eight transverse coordinates z_k and $y_{k'}$ are rescaled according to

$$z_k \to z_k/R \qquad y_{k'} \to y_{k'}/R , \qquad (2.4)$$

and p_- is held fixed while R and J are taken to be infinite. Under the AdS/CFT parameter duality, p_- is given by

$$p_- = \frac{1}{\sqrt{\lambda'}} = \frac{J}{\sqrt{g^2_{\rm YM} N_c}} , \qquad (2.5)$$

where λ' is the so-called modified 't Hooft coupling. Keeping the first $1/R^2$ (equivalently $1/J$) correction to this limit, the metric becomes

$$\begin{aligned} ds^2 &= 2\,dx^+dx^- - (x^A)^2(dx^+)^2 + (dx^A)^2 \\ &\quad + \frac{1}{R^2}\left[-2y^2 dx^+ dx^- + \frac{1}{2}(y^4 - z^4)(dx^+)^2 + (dx^-)^2 + \frac{1}{2}z^2 dz^2 - \frac{1}{2}y^2 dy^2 \right] \\ &\quad + O(R^{-4}) , \end{aligned} \qquad (2.6)$$

where the coordinates x^A $(A = 1, \ldots, 8)$ span the transverse $SO(8)$ subspace.

The complete IIB superstring theory in this background can be formulated in terms of the Green-Schwarz action built from supersymmetric Cartan one-forms and superconnections on the coset space associated with $AdS_5 \times S^5$ [20, 21, 22, 23, 24]. The complete Green-Schwarz superstring Lagrangian, which is dependent on the scale radius R, can then be expanded in large R to extract the pp-wave limit of the theory plus higher-order corrections in R^{-2}.

At leading order in this expansion the Hamiltonian $H_{\rm pp}$ consists of a free theory of eight massive bosons and fermions:

$$H_{\rm pp} = \frac{p_-}{2}(x^A)^2 + \frac{1}{2p_-}\left[(p_A)^2 + (x'^A)^2\right] + i\rho\Pi\psi + \frac{i}{2}\psi\psi' - \frac{i}{2p_-^2}\rho\rho' \ . \tag{2.7}$$

The fields p_A are bosonic momenta conjugate to x^A, ψ denote fermionic fields with conjugate variables ρ, and the shorthand notation x'^A denotes the worldsheet derivative $\partial_\sigma x^A$. The fermion fields ψ_α are eight-component complex spinors constructed from two $SO(9,1)$ Majorana-Weyl spinors of equal chirality, and the matrix Π is defined in terms of the eight-dimensional $SO(8)$ gamma matrices γ^a, $\bar\gamma^a$ as $\Pi \equiv \gamma^1\bar\gamma^2\gamma^3\bar\gamma^4$. (For further details the reader is referred to [10, 11].)

The first curvature correction to the background metric gives rise to an interaction Hamiltonian denoted by $H_{\rm int}$, which, in turn, is broken into bosonic ($H_{\rm BB}$), fermionic ($H_{\rm FF}$) and bose-fermi mixed ($H_{\rm BF}$) sectors:

$$H = H_{\rm pp} + H_{\rm int} \qquad H_{\rm int} = H_{\rm BB} + H_{\rm FF} + H_{\rm BF} \ . \tag{2.8}$$

The complete perturbation $H_{\rm int}$ was computed explicitly in terms of the constituent fields described above in [10, 11]. Since we will deal mostly with the bosonic sector of the theory in the present study, we state $H_{\rm BB}$ explicitly but refer the reader to [10, 11] for the detailed form of the remaining sectors $H_{\rm BF}$ and $H_{\rm FF}$:

$$\begin{aligned} H_{\rm BB} = {} & \frac{1}{R^2}\Bigg\{\frac{1}{4p_-}\left[-y^2\left(p_z^2 + z'^2 + 2y'^2\right) + z^2\left(p_y^2 + y'^2 + 2z'^2\right)\right] + \frac{p_-}{8}\left[(x^A)^2\right]^2 \\ & -\frac{1}{8p_-^3}\left\{\left[(p_A)^2\right]^2 + 2(p_A)^2(x'^A)^2 + \left[(x'^A)^2\right]^2\right\} + \frac{1}{2p_-^3}\left(x'^A p_A\right)^2\Bigg\} \ . \end{aligned} \tag{2.9}$$

The $SO(8)$ indices $(A, B, C, \ldots = 1, \ldots, 8)$ are generally split into AdS_5 and S^5 subspaces using the lower-case Latin notation $a, b, c, \ldots = 1, \ldots, 4$ (for $SO(4)_{AdS}$) and $a', b', c', \ldots = 5, \ldots, 8$ (for $SO(4)_{S^5}$). The Greek indices $\alpha, \beta, \gamma, \ldots = 1, \ldots, 8$ are used to label the eight components of the fermionic fields ψ, ρ. We point out that the interaction Hamiltonian is strictly quartic in fields, a fact which will be important in the subsequent analysis.

The vacuum state carries the S^5 string angular momentum J and is labelled by $|J\rangle$; the complete Fock space of string states is generated by acting on $|J\rangle$ with any number of the creation operators $a_n^{A\dagger}$ (bosonic) and $b_n^{\alpha\dagger}$ (fermionic), where the lower indices $n, m, l, \ldots$ denote mode numbers. The excitation number of string states (defined by the number of creation oscillators acting on the ground state) will also be referred to as the impurity number, and string states with a total of $N_B + N_F = N$ impurities will contain N_B bosonic and N_F fermionic impurities:

$$|N_B, N_F; J\rangle \equiv \underbrace{a_{n_1}^{A_1\dagger} a_{n_2}^{A_2\dagger} \ldots a_{n_{N_B}}^{A_{N_B}\dagger}}_{N_B} \underbrace{b_{n_1}^{\alpha_1\dagger} b_{n_2}^{\alpha_2\dagger} \ldots b_{n_{N_F}}^{\alpha_{N_F}\dagger}}_{N_F} |J\rangle \ . \tag{2.10}$$

States constructed in this manner fall into two disjoint subsectors populated by spacetime bosons (N_F even) and spacetime fermions (N_F odd). In this notation the pure-boson states $|N_B, 0; J\rangle$ are mixed only by $H_{\rm BB}$ and the pure-fermion states $|0, N_F; J\rangle$ are acted on by $H_{\rm FF}$. The more general spacetime-boson states $|N_B, \text{even}; J\rangle$ are acted on by the complete interaction Hamiltonian $H_{\rm int}$, as are the spacetime-fermion states $|N_B, \text{odd}; J\rangle$. There is of course no mixing between spacetime bosons and fermions; this block-diagonalization is given schematically in table 1.

$H_{\rm int}$	$\|N_B, 0; J\rangle$	$\|N_B, \text{even}; J\rangle$	$\|N_B, \text{odd}; J\rangle$	$\|0, \text{odd}; J\rangle$
$\langle N_B, 0; J\|$	$H_{\rm BB}$	$H_{\rm BF}$		
$\langle N_B, \text{even}; J\|$	$H_{\rm BF}$	$H_{\rm BB} + H_{\rm BF} + H_{\rm FF}$		
$\langle N_B, \text{odd}; J\|$			$H_{\rm BB} + H_{\rm BF} + H_{\rm FF}$	$H_{\rm BF}$
$\langle 0, \text{odd}; J\|$			$H_{\rm BF}$	$H_{\rm FF}$

Table 1: Interaction Hamiltonian on N-impurity string states ($N_B + N_F = N$)

The full interaction Hamiltonian can be further block-diagonalized by projecting onto certain protected sectors of string states, and we will focus in this study on three such sectors. Two of these sectors are spanned by purely bosonic states $|N_B, 0; J\rangle$ projected onto symmetric-traceless irreps in either the $SO(4)_{AdS}$ or $SO(4)_{S^5}$ subspaces. Another sector which is known to decouple at all orders in λ' is comprised of purely fermionic states $|0, N_F; J\rangle$ projected onto either of two subspaces of $SO(4) \times SO(4)$ labelled, in an $SU(2)^2 \times SU(2)^2$ notation, by $(\mathbf{2}, \mathbf{1}; \mathbf{2}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2}; \mathbf{1}, \mathbf{2})$, and symmetrized in spinor indices. Each of these sectors can also be labelled by the subalgebra of the full superconformal algebra that corresponds to the symmetry under which they are invariant. The bosonic $SO(4)_{AdS}$ and $SO(4)_{S^5}$ sectors are labelled by $\mathfrak{sl}(2)$ and $\mathfrak{su}(2)$ subalgebras, respectively, while the two fermionic sectors fall into $\mathfrak{su}(1|1)$ subsectors of the closed $\mathfrak{su}(2|3)$ system studied in [6, 15, 25].

3 N-Impurity string energy spectra

In the large-J expansion about the free pp-wave theory, we will isolate $O(J^{-1})$ corrections to the energy eigenvalues of N-impurity string states according to

$$E(\{q_j\}, N, J) = \sum_{j=1}^{N} \sqrt{1 + q_j^2 \lambda'} + \delta E(\{q_j\}, N, J) + O(J^{-2}) \ . \tag{3.1}$$

The spectrum is generically dependent upon λ', J and the mode numbers $\{n_j\}, \{q_j\}, \ldots$, where j is understood to label either the complete set of impurities ($j = 1, \ldots, N$) or some subset thereof (eg. $j = 1, \ldots, N_F$). The leading order term in this expansion is the N-impurity free energy of states on the pp-wave geometry, and $\delta E(\{q_j\}, N, J)$ always enters

at $O(J^{-1})$. When it becomes necessary, we will also expand the $O(1/J)$ energy shift in the small-λ' loop expansion:

$$\delta E(\{q_j\}, N, J) = \sum_{i=1}^{\infty} \delta E^{(i)}(\{q_j\}, N, J)(\lambda')^i \ . \tag{3.2}$$

Finding the explicit form of $\delta E(\{q_j\}, N, J)$ for N-impurity string states in certain interesting sectors of the theory will be our primary goal. As a side result, however, we will see that the spectrum of *all* states in the theory will be determined to two-loop order in λ' by the specific eigenvalues we intend to compute.

We begin by noting that the canonical commutation relations of the bosonic fields x^A and p_A allow us to expand $H_{\rm BB}$ in bosonic creation and annihilation operators using

$$\begin{aligned} x^A(\sigma,\tau) &= \sum_{n=-\infty}^{\infty} x_n^A(\tau) e^{-ik_n\sigma} \\ x_n^A(\tau) &= \frac{i}{\sqrt{2\omega_n}}\left(a_n^A e^{-i\omega_n\tau} - a_{-n}^{A\dagger} e^{i\omega_n\tau}\right) \ , \end{aligned} \tag{3.3}$$

where $k_n = n$ are integer-valued, $\omega_n = \sqrt{p_-^2 + n^2}$ and the operators a_n^A and $a_n^{A\dagger}$ obey the usual relation $\left[a_m^A, a_n^{B\dagger}\right] = \delta_{mn}\delta^{AB}$. Since we are only interested in computing diagonal matrix elements of $H_{\rm BB}$ between physical string states with equal numbers of excitations, we can restrict the oscillator expansion to contain only equal numbers of creation and annihilation operators (all other combinations automatically annihilate between equal-impurity string states). Explicitly, we obtain the following expansion:

$$\begin{aligned} H_{\rm BB} &= -\frac{1}{32p_-R^2}\sum \frac{\delta(n+m+l+p)}{\xi} \times \\ &\Bigg\{2\Big[\xi^2 - (1 - k_lk_pk_nk_m) + \omega_n\omega_mk_lk_p + \omega_l\omega_pk_nk_m + 2\omega_n\omega_lk_mk_p \\ &+2\omega_m\omega_pk_nk_l\Big]a_{-n}^{\dagger A}a_{-m}^{\dagger A}a_l^Ba_p^B + 4\Big[\xi^2 - (1 - k_lk_pk_nk_m) - 2\omega_n\omega_mk_lk_p + \omega_l\omega_mk_nk_p \\ &-\omega_n\omega_lk_mk_p - \omega_m\omega_pk_nk_l + \omega_n\omega_pk_mk_l\Big]a_{-n}^{\dagger A}a_{-l}^{\dagger B}a_m^Aa_p^B + 2\Big[8k_lk_pa_{-n}^{\dagger i}a_{-l}^{\dagger j}a_m^ia_p^j \\ &+2(k_lk_p + k_nk_m)a_{-n}^{\dagger i}a_{-m}^{\dagger i}a_l^ja_p^j + (\omega_l\omega_p + k_lk_p - \omega_n\omega_m - k_nk_m)a_{-n}^{\dagger i}a_{-m}^{\dagger i}a_l^{j'}a_p^{j'} \\ &-4(\omega_l\omega_p - k_lk_p)a_{-n}^{\dagger i}a_{-l}^{\dagger j'}a_m^ia_p^{j'} - (i,j \rightleftharpoons i',j')\Big]\Bigg\} \ , \end{aligned} \tag{3.4}$$

where $\xi \equiv \sqrt{\omega_n\omega_m\omega_l\omega_p}$.

3.1 The $SO(4)_{S^5}$ ($\mathfrak{su}(2)$) sector

We begin in the $\mathfrak{su}(2)$ sector spanned by symmetric-traceless pure-boson states excited in the S^5 subspace. Because we are restricting our attention to $SO(4)_{S^5}$ states symmetric in

their vector indices, we form the following oscillators:

$$a_n = \frac{1}{\sqrt{2}}\left(a_n^5 + i a_n^6\right) \qquad \bar{a}_n = \frac{1}{\sqrt{2}}\left(a_n^5 - i a_n^6\right) \ . \tag{3.5}$$

By taking matrix elements of the form

$$\langle J|\, a_{n_1} a_{n_2} \ldots a_{n_{N_B}} (H_{\rm BB}) a^\dagger_{n_1} a^\dagger_{n_2} \ldots a^\dagger_{n_{N_B}} \,|J\rangle \ , \tag{3.6}$$

we can therefore select out excitations in the $(5,6)$-plane of the S^5 subspace and make the symmetric-traceless projection manifest. (More generally we can project onto any (n,m)-plane, as long as $n \neq m$ and both are chosen to lie in the S^5 subspace.)

There are two basic oscillator structures of $H_{\rm BB}$ in eqn. (3.4): one in which the creation (annihilation) operators are contracted in their $SO(4) \times SO(4)$ indices

$$a^{\dagger A}_{-n} a^{\dagger A}_{-m} a^B_l a^B_p \ ,$$

and one where pairs of creation and annihilation operators are contracted

$$a^{\dagger A}_{-n} a^{\dagger B}_{-l} a^A_m a^B_p \ .$$

In terms of the a_n and $\bar{a}_n$ fields of eqn. (3.5), the former structure contains

$$a^{\dagger A}_{-n} a^{\dagger A}_{-m} a^B_l a^B_p \Big|_{(5,6)} = \left(a^\dagger_{-n}\, \bar{a}^\dagger_{-m} + \bar{a}^\dagger_{-n}\, a^\dagger_{-m}\right)\left(a_l\, \bar{a}_p + \bar{a}_l\, a_p\right) \ , \tag{3.7}$$

which cannot contribute to $\mathfrak{su}(2)$ matrix elements of the form appearing in (3.6). The latter structure, however, contains

$$a^{\dagger A}_{-n} a^{\dagger B}_{-l} a^A_m a^B_p \Big|_{(5,6)} = \bar{a}^\dagger_{-n}\, \bar{a}^\dagger_{-l}\, \bar{a}_m\, \bar{a}_p + a^\dagger_{-n}\, a^\dagger_{-l}\, a_m\, a_p \ , \tag{3.8}$$

which will contribute to the $\mathfrak{su}(2)$ energy spectrum.

The string states appearing in the matrix element of eqn. (3.6) have been written in the generic form

$$a^\dagger_{n_1} a^\dagger_{n_2} \ldots a^\dagger_{n_{N_B}} \,|J\rangle \ ,$$

and, as usual, they are subject to the level-matching condition

$$\sum_{j=1}^{N_B} n_j = 0 \ . \tag{3.9}$$

The complete set of mode indices $\{n_1, n_2, \ldots, n_{N_B}\}$ can contain one or more subsets of indices that are equal, while still satisfying eqn. (3.9); this scenario complicates the calculation of energy eigenvalues to some extent. We will eventually compute the eigenvalues of interest

for completely general string states, but for purposes of illustration and to introduce our strategy we will start with the simplest case in which no two mode numbers are equal ($n_1 \neq n_2 \neq \ldots \neq n_{N_B}$). To organize the presentation, we will generally use mode numbers labelled by $\{n_j\}$ to denote those which are inequivalent from each other, while $\{q_j\}$ will be allowed to overlap. Between states with completely distinct mode indices, the oscillator structure in eqn. (3.8) exhibits the following matrix element:

$$\begin{aligned}\langle J|\, a_{n_1} a_{n_2} \ldots a_{N_B} (a^\dagger_{-n} a^\dagger_{-l} a_m a_p) a^\dagger_{n_1} a^\dagger_{n_2} \ldots a^\dagger_{N_B} |J\rangle \\ = \frac{1}{2} \sum_{\substack{j,k=1 \\ j\neq k}}^{N_B} \Big(\delta_{n_j+n}\, \delta_{n_k+l}\, \delta_{n_j-m}\, \delta_{n_k-p} + \delta_{n_j+n}\, \delta_{n_k+l}\, \delta_{n_k-m}\, \delta_{n_j-p} \\ + \delta_{n_j+l}\, \delta_{n_k+n}\, \delta_{n_j-m}\, \delta_{n_k-p} + \delta_{n_j+l}\, \delta_{n_k+n}\, \delta_{n_k-m}\, \delta_{n_j-p} \Big) \ . \quad (3.10)\end{aligned}$$

With this in hand, it is a straightforward exercise to compute the energy eigenvalue of the $SO(4)_{S^5}$ bosonic interaction Hamiltonian in the N_B-impurity symmetric-traceless irrep (with unequal mode indices): we simply attach the $H_{\rm BB}$ coefficient of the oscillator structure $a^\dagger_{-n} a^\dagger_{-l} a_m a_p$ to the right-hand side of eqn. (3.10) and carry out the summation over mode numbers. The result is remarkably compact:

$$\delta E_{S^5}(\{n_i\}, N_B, J) = -\frac{1}{J} \sum_{\substack{j,k=1 \\ j\neq k}}^{N_B} \frac{1}{2\,\omega_{n_j}\omega_{n_k}} \left[n_k^2 + n_j^2 \left(1 + n_k^2 \lambda'\right) + n_j n_k \left(1 - \omega_{n_j}\omega_{n_k}\lambda'\right) \right] \ . \quad (3.11)$$

This $\mathfrak{su}(2)$ formula can be checked against previously obtained string theory results in the two- and three-impurity regimes. Namely, the two-impurity eigenvalue computed in [10, 11] takes the form (which is exact in λ')

$$\delta E_{S^5}(n_1, n_2, J) = -\frac{2\, n_1^2 \lambda'}{J} \ , \quad (3.12)$$

where we have set $n_2 = -n_1$ using eqn. (3.9). This eigenvalue matches the general formula in eqn. (3.11) restricted to two impurities. The $\mathfrak{su}(2)$ eigenvalue for three impurities with unequal mode indices ($n_1 \neq n_2 \neq n_3$) was calculated in [12] and found to be

$$\begin{aligned}\delta E_{S^5}(n_1, n_2, n_3, J) \ &= \ -\frac{1}{J\omega_{n_1}\omega_{n_2}\omega_{n_3}} \Big\{ \left[n_1 n_2 + n_2^2 + n_1^2(1 + n_2^2\lambda') \right] \omega_{n_3} \\ &+ \left[n_1 n_3 + n_3^2 + n_1^2(1 + n_3^2\lambda') \right] \omega_{n_2} + \left[n_2 n_3 + n_3^2 + n_2^2(1 + n_3^2\lambda') \right] \omega_{n_1} \\ &- \left[n_2 n_3 + n_1(n_2 + n_3) \right] \lambda' \omega_{n_1}\omega_{n_2}\omega_{n_3} \Big\} \ . \quad (3.13)\end{aligned}$$

It is also easy to check that eqn. (3.11) reproduces this formula exactly for $N_B = 3$.

Since eqn. (3.11) matches all previously computed results from the string theory in this sector, it must therefore agree with corresponding $\mathfrak{su}(2)$ gauge theory predictions only to

two-loop order in λ. We note, however, that eqn. (3.11) is *identical* to the N-impurity $O(J^{-1})$ energy shift (with unequal mode numbers) obtained from the $\mathfrak{su}(2)$ string Bethe ansatz of [19].

To treat the slightly more complicated scenario of overlapping mode indices (which can occur for three or more impurities), we introduce the normalized eigenvectors

$$\frac{1}{\sqrt{N_q!}}\left(a_q^\dagger\right)^{N_q} a_{n_1}^\dagger a_{n_2}^\dagger \dots a_{n_{(N_B-N_q)}}^\dagger \,|J\rangle \ , \tag{3.14}$$

which contain a single subset of N_q bosonic oscillators $a_q^\dagger$ which all share the same mode index q. The remaining indices $n_i \in \{n_1, n_2, \dots, n_{N_B-N_q}\}$ are all separate from q and unequal from each other, such that the level-matching condition in eqn. (3.9) now reads

$$N_q\, q + \sum_{j=1}^{N_B-N_q} n_j = 0 \ . \tag{3.15}$$

For this case we compute a matrix element analogous to that in eqn. (3.10):

$$\begin{aligned}
&\frac{1}{N_q!}\,\langle J|\,(a_q)^{N_q}\, a_{n_1}a_{n_2}\dots a_{n_{(N_B-N_q)}}(a_{-n}^\dagger a_{-l}^\dagger a_m a_p)\left(a_q^\dagger\right)^{N_q} a_{n_1}^\dagger a_{n_2}^\dagger \dots a_{n_{(N_B-N_q)}}^\dagger\,|J\rangle \\
&\quad = N_q(N_q-1)\delta_{p-q}\,\delta_{m-q}\,\delta_{n+q}\,\delta_{l+q} + L\sum_{j=1}^{N_B-N_q}\Big(\delta_{p-q}\,\delta_{n+q}\,\delta_{m-n_j}\,\delta_{l+n_j} + \delta_{m-q}\,\delta_{n+q}\,\delta_{p-n_j}\,\delta_{l+n_j} \\
&\quad +\delta_{p-q}\,\delta_{l+q}\,\delta_{m-n_j}\,\delta_{n+n_j} + \delta_{m-q}\,\delta_{l+q}\,\delta_{p-n_j}\,\delta_{n+n_j}\Big) + \frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{N_B-N_q}\Big(\delta_{n_j+n}\,\delta_{n_k+l}\,\delta_{n_j-m}\,\delta_{n_k-p} \\
&\quad +\delta_{n_j+n}\,\delta_{n_k+l}\,\delta_{n_k-m}\,\delta_{n_j-p} + \delta_{n_j+l}\,\delta_{n_k+n}\,\delta_{n_j-m}\,\delta_{n_k-p} + \delta_{n_j+l}\,\delta_{n_k+n}\,\delta_{n_k-m}\,\delta_{n_j-p}\Big) \ .
\end{aligned} \tag{3.16}$$

Using this result, we arrive at the $\mathfrak{su}(2)$ energy shift for string states with N_B total excitations containing an N_q-component subset of oscillators that share the same mode index q:

$$\begin{aligned}
\delta E_{S^5}(\{n_i\}, q, N_q, N_B, J) \;&=\; -\frac{N_q(N_q-1)q^2}{2J\omega_q^2} \\
&\quad - \sum_{j=1}^{N_B-N_q}\frac{N_q}{J\omega_q\omega_{n_j}}\left[q^2 + n_j^2(1+q^2\lambda') + q\,n_j\left(1-\omega_q\omega_{n_j}\lambda'\right)\right] \\
&\quad - \sum_{\substack{j,k=1\\ j\neq k}}^{N_B-N_q}\frac{1}{2J\,\omega_j\omega_k}\left[n_k^2 + n_j^2\left(1+n_k^2\lambda'\right) + n_j n_k\left(1-\omega_j\omega_k\lambda'\right)\right] \ .
\end{aligned} \tag{3.17}$$

This formula can be compared with the three-impurity $\mathfrak{su}(2)$ energy shift with two equal mode indices ($N_q = 2$) obtained in [12]. For this particular case we can set the isolated mode

number to $-2q$ using the level-matching condition to simplify the result:

$$\delta E_{S^5}(q,J) = -\frac{q^2}{J\omega_q^2\omega_{2\,q}}\left[\omega_{2\,q}\left(5+4\,q^2\lambda'\right)+\omega_q\left(6+8\,q^2\lambda'\right)\right] \ . \tag{3.18}$$

It is easy to show that eqn. (3.17) exactly reproduces this energy shift when restricted to $N_B = 3$ with a subset of two mode numbers equal to q.

We now generalize the analysis completely by using eigenstates with M mode-index subsets, where all mode indices are equal within these subsets:

$$\frac{\left(a^\dagger_{q_1}\right)^{N_{q_1}}}{\sqrt{N_{q_1}!}}\frac{\left(a^\dagger_{q_2}\right)^{N_{q_2}}}{\sqrt{N_{q_2}!}}\cdots\frac{\left(a^\dagger_{q_M}\right)^{N_{q_M}}}{\sqrt{N_{q_M}!}}\,|J\rangle \ .$$

The j^{th} subset contains N_{q_j} oscillators with equal mode index q_j, and the total impurity number is again N_B, such that

$$\sum_{i=1}^{M} N_{q_i} = N_B \qquad \sum_{i=1}^{M} N_{q_i} q_i = 0 \ . \tag{3.19}$$

The matrix element of $a^\dagger_{-n}\,a^\dagger_{-l}\,a_m\,a_p$ between the above states, analogous to eqns. (3.10,3.16), is

$$\begin{aligned}
&\langle J|\,\frac{\left(a_{q_1}\right)^{N_{q_1}}}{\sqrt{N_{q_1}!}}\cdots\frac{\left(a_{q_M}\right)^{N_{q_M}}}{\sqrt{N_{q_M}!}}\left(a^\dagger_{-n}\,a^\dagger_{-l}\,a_m\,a_p\right)\frac{\left(a^\dagger_{q_1}\right)^{N_{q_1}}}{\sqrt{N_{q_1}!}}\cdots\frac{\left(a^\dagger_{q_M}\right)^{N_{q_M}}}{\sqrt{N_{q_M}!}}\,|J\rangle \\
&\quad = \sum_{j=1}^{M} N_{q_j}(N_{q_j}-1)\,\delta_{n+n_j}\,\delta_{l+n_j}\,\delta_{m-n_j}\,\delta_{p-n_j} + \frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{M} N_{q_j}N_{q_k}\Big(\delta_{n+n_k}\,\delta_{l+n_j}\,\delta_{m-n_k}\,\delta_{p-n_j} \\
&\quad +\delta_{n+n_j}\,\delta_{l+n_k}\,\delta_{m-n_k}\,\delta_{p-n_j} + \delta_{n+n_k}\,\delta_{l+n_j}\,\delta_{m-n_j}\,\delta_{p-n_k} + \delta_{n+n_j}\,\delta_{l+n_k}\,\delta_{m-n_j}\,\delta_{p-n_k}\Big) \ .
\end{aligned} \tag{3.20}$$

We thereby obtain the completely general $\mathfrak{su}(2)$ energy shift for N_B-impurity string states containing M equal-mode-index subsets of oscillators:

$$\begin{aligned}
\delta E_{S^5}(\{q_i\},\{N_{q_i}\},M,J) \ &= \ -\frac{1}{2J}\Bigg\{\sum_{j=1}^{M} N_{q_j}(N_{q_j}-1)\left(1-\frac{1}{\omega_{q_j}^2\lambda'}\right) \\
&\quad + \sum_{\substack{j,k=1\\ j\neq k}}^{M}\frac{N_{q_j}N_{q_k}}{\omega_{q_j}\omega_{q_k}}\left[q_k^2+q_j^2\omega_{q_k}^2\lambda'+q_jq_k(1-\omega_{q_j}\omega_{q_k}\lambda')\right]\Bigg\} \ .
\end{aligned} \tag{3.21}$$

This master formula can be used to determine the $\mathfrak{su}(2)$ string energy spectrum to $O(J^{-1})$ for all possible physical string states in this sector.

By taking $M = 2$ and setting $N_{n_1} = N_{n_2} = 1$ (using the unequal mode indices $\{n_1, n_2\}$), we recover from this equation the exact two-impurity result recorded in eqn. (3.12) above,

with $n_2 = -n_1$. For $M = 3$ and $N_{n_1} = N_{n_2} = N_{n_3} = 1$, we get the complete three-impurity unequal-mode-number ($n_1 \neq n_2 \neq n_3$) formula found in eqn. (3.13). Finally, the three-impurity eigenvalue with two equal mode indices ($q_1 = q_2,\ q_3 = -2q_1$) given in eqn. (3.18) can also be extracted from eqn. (3.21) by setting $M = 2$, $N_{q_1} = 2$ and $N_{q_2} = 1$.

We also note that eqn. (3.21) agrees perfectly with the corresponding near-pp-wave formula derived from the $\mathfrak{su}(2)$ string Bethe ansatz of [19] for completely general mode-number assignment. This successful match stands as very strong evidence that their ansatz is correct, at least to $O(J^{-1})$.

3.2 The $SO(4)_{AdS}$ ($\mathfrak{sl}(2)$) sector

Following the derivation of eqn. (3.21) for the energy eigenvalues of arbitrary string states in the symmetric-traceless $SO(4)_{S^5}$ sector, it is straightforward to find the analogous expression for symmetric-traceless string states excited in the $SO(4)_{AdS}$ subspace, dual to operators in the $\mathfrak{sl}(2)$ sector of the corresponding gauge theory. We can define, for example,

$$a_n = \frac{1}{\sqrt{2}}\left(a_n^1 + i a_n^2\right) \qquad \bar{a}_n = \frac{1}{\sqrt{2}}\left(a_n^1 - i a_n^2\right)\ , \tag{3.22}$$

and carry out the above calculations by computing general matrix elements of $a_{-n}^\dagger a_{-l}^\dagger a_m a_p$ defined in terms of these oscillators. (Here we can project onto any (n,m)-plane in the AdS_5 subspace, as long as $n \neq m$.) General string energy eigenvalues in the $SO(4)_{AdS}$ symmetric-traceless irrep are thus found to be

$$\begin{aligned}\delta E_{AdS}(\{q_i\},\{N_{q_i}\},M,J) \;&=\; \frac{1}{2J}\Bigg\{\sum_{j=1}^{M} N_{q_j}(N_{q_j}-1)\left(1-\frac{1}{\omega_{q_j}^2\lambda'}\right)\\ &\quad + \sum_{\substack{j,k=1\\ j\neq k}}^{M}\frac{N_{q_j}N_{q_k}}{\omega_{q_j}\omega_{q_k}} q_j q_k\left[1 - q_j q_k\lambda' + \omega_{q_j}\omega_{q_k}\lambda'\right]\Bigg\}\ .\end{aligned} \tag{3.23}$$

For later reference we record the limit of this equation for states with completely unequal mode indices ($\{N_{n_i}\} = 1,\ M = N_B$):

$$\delta E_{AdS}(\{n_i\},N_B,J) \;=\; \frac{1}{2J}\sum_{\substack{j,k=1\\ j\neq k}}^{N_B}\frac{n_j n_k}{\omega_{n_j}\omega_{n_k}}\left[1 - n_j n_k\lambda' + \omega_{n_j}\omega_{n_k}\lambda'\right]\ . \tag{3.24}$$

When $M = 2$ and $N_{n_1} = N_{n_2} = 1$ in eqn. (3.24), we find the two-impurity eigenvalue (with $n_2 = -n_1$)

$$\delta E_{AdS}(n_1,J) = -\frac{2\,n_1^2\lambda'}{J}\ , \tag{3.25}$$

which agrees with the two-impurity result reported in [10, 11] (the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ eigenvalues are degenerate in the two-impurity regime). For the three-impurity eigenvalue with three unequal mode indices we set $M = 3$ and $N_{n_1} = N_{n_2} = N_{n_3} = 1$ to obtain

$$\begin{aligned}\delta E_{AdS}(n_1, n_2, n_3, J) &= \frac{1}{J\omega_{n_1}\omega_{n_2}\omega_{n_3}}\Big\{ n_1 n_3(1 - n_1 n_3\lambda')\,\omega_{n_2} + n_1 n_2(1 - n_1 n_2\lambda')\,\omega_{n_3} \\ &\quad + n_2 n_3(1 - n_2 n_3\lambda')\,\omega_{n_1} + [n_1 n_2 + n_3(n_1 + n_2)]\,\lambda'\omega_{n_1}\omega_{n_2}\omega_{n_3}\Big\}\ , \end{aligned} \tag{3.26}$$

which precisely reproduces the corresponding $\mathfrak{sl}(2)$ result reported in [12]. Finally, by setting $M = 2$, $N_{q_1} = 2$, $N_{q_2} = 1$ and $q_1 = q_2 = q$, $q_3 = -2q$, eqn. (3.23) provides the following three-impurity eigenvalue with two equal mode indices:

$$\delta E_{AdS}(q, J) = -\frac{q^2}{J\omega_q^2\omega_{2\,q}}\left[\omega_{2\,q}\left(3 + 4\,q^2\lambda'\right) + \omega_q\left(4 + 8\,q^2\lambda'\right)\right]\ . \tag{3.27}$$

This again matches the three-impurity formula found in [12].

3.3 The $\mathfrak{su}(1|1)$ sector

Based on the above results in the bosonic $SO(4)_{AdS}$ and $SO(4)_{S^5}$ symmetric-traceless sectors, we can easily formulate a conjecture for the N-impurity eigenvalue of symmetrized pure-fermion states in either the $(\mathbf{2}, \mathbf{1}; \mathbf{2}, \mathbf{1})$ or $(\mathbf{1}, \mathbf{2}; \mathbf{1}, \mathbf{2})$ of $SO(4) \times SO(4)$, labelled by the $\mathfrak{su}(1|1)$ subalgebra. We first note that, since these states are composed of fermionic oscillators which are symmetrized in their spinor indices, no states in this sector can carry subsets of overlapping mode numbers (since they would automatically vanish). Furthermore, when restricting to states with completely unequal mode indices, we can see that the N-impurity eigenvalues obtained for the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ sectors (eqns. (3.11) and (3.24)) are obvious generalizations of the corresponding three-impurity formulas (eqns. (3.13) and (3.26), respectively). Namely, if the three-impurity eigenvalues take the generic form

$$\delta E(n_1, n_2, n_3, J) = \sum_{\substack{j,k=1 \\ j\neq k}}^{3} F(n_j, n_k)\ , \tag{3.28}$$

the N-impurity generalization is simply

$$\delta E(\{n_i\}, N, J) = \sum_{\substack{j,k=1 \\ j\neq k}}^{N} F(n_j, n_k)\ . \tag{3.29}$$

By carrying this over to the $\mathfrak{su}(1|1)$ sector, we find the N-impurity eigenvalue of $H_{\rm FF}$ between symmetrized $(\mathbf{2}, \mathbf{1}; \mathbf{2}, \mathbf{1})$ or $(\mathbf{1}, \mathbf{2}; \mathbf{1}, \mathbf{2})$ fermions (the eigenvalues of both are necessarily

degenerate):

$$\delta E_{\mathfrak{su}(1|1)}(\{n_i\}, N_F, J) = -\frac{1}{4J}\sum_{\substack{j,k=1\\ j\neq k}}^{N_F}\frac{1}{\omega_{n_j}\omega_{n_k}}\left[n_j^2 + n_k^2 + 2n_j^2 n_k^2\lambda' - 2\,n_j n_k\omega_{n_j}\omega_{n_k}\lambda'\right] \ . \quad (3.30)$$

For $N_F = 2$, this formula matches the two-impurity result in [10, 11]:

$$\delta E_{\mathfrak{su}(1|1)}(n_1, J) = -\frac{2\,n_1^2\lambda'}{J} \ , \quad (3.31)$$

with $n_2 = -n_1$ (this eigenvalue overlaps with the corresponding two-impurity $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ values). When $N_F = 3$ we of course recover the three-impurity eigenvalue reported in [12]:

$$\begin{aligned}\delta E_{\mathfrak{su}(1|1)}(n_1, n_2, n_3, J) \ &= \ -\frac{1}{4\,J\omega_{n_1}\omega_{n_2}\omega_{n_3}}\Big\{-4\left(n_2 n_3 + n_1(n_2+n_3)\right)\lambda'\omega_{n_1}\omega_{n_2}\omega_{n_3} \\ &+\Big[\omega_{n_1}\left(2\,n_3^2 + 4\,n_2^2 n_3^2\lambda' + 2\,n_2^2\right) + \left(n_3\to n_2,\ n_2\to n_1,\ n_1\to n_3\right) + \left(n_1 \rightleftharpoons n_2\right)\Big]\Big\} \ .\end{aligned} \quad (3.32)$$

It would be straightforward to check eqn. (3.30) against an explicit four-impurity calculation in the string theory, for example. Better yet, one might carry out the direct N-impurity calculation in the $H_{\rm FF}$ sector analogous to the above calculations for $H_{\rm BB}$. The latter would certainly be more technically complicated than in the bosonic sectors, and for the moment we leave eqn. (3.30) as it stands, withholding direct verification for a future study.

4 Spectral decomposition

At one- and two-loop order in λ' we can infer from basic arguments the spectral decomposition of the extended N-impurity superconformal multiplet of $O(J^{-1})$ energy corrections to the pp-wave limit. For simplicity we will restrict the discussion to eigensystems with completely unequal mode numbers, though the generalization to more complicated cases is straightforward. To begin we will review the two- and three-impurity supermultiplet structures studied in [10, 11, 12].

We denote the one- and two-loop energy eigenvalue shifts as $\Lambda^{(1)}$ and $\Lambda^{(2)}$, according to the generic formula

$$\begin{aligned}E(\{n_j\}, N, J) \ &= \ N + \frac{\lambda'}{2}\sum_{j=1}^{N} n_j^2\left(1 + \frac{\Lambda^{(1)}}{J} + O(J^{-2})\right) \\ &\quad -\frac{\lambda'^2}{4}\sum_{j=1}^{N} n_j^4\left(\frac{1}{2} + \frac{\Lambda^{(2)}}{J} + O(J^{-2})\right) + O(\lambda'^3) \ . \end{aligned} \quad (4.1)$$

The fact that these energy shifts can be expressed as coefficients of $\sum n_j^2$ and $\sum n_j^4$ is not obvious. In the two- and three-impurity cases this was shown to be true by direct diagonalization of the Hamiltonian. By expanding eqns. (3.21,3.23,3.30) in small λ', it can also be seen that the more general N-impurity $\mathfrak{su}(2)$, $\mathfrak{sl}(2)$ and $\mathfrak{su}(1|1)$ eigenvalues adhere to this structure to two-loop order. We will argue that the remaining energy shifts (those in non-protected subsectors) can be obtained from the protected sectors through half-integer shifts of the S^5 angular momentum J: it will therefore be seen that all energies considered here will appear in the form given in eqn. (4.1).

As described in [12], the conformal invariance of the full $\mathfrak{psu}(2,2|4)$ symmetry algebra of the theory guarantees that the energy eigenvalues (and hence $\Lambda^{(1)}$ and $\Lambda^{(2)}$) will be organized into conformal (sub)multiplets built on conformal primary (or highest weight) states. Within a given submultiplet we refer to states with lowest energy as super-primary states, and the other conformal primaries within the submultiplet are obtained by acting on super-primaries with any of the eight supercharges, labelled by $\mathcal{Q}_\alpha$, that increment $\Lambda^{(1)}$ or $\Lambda^{(2)}$ by a fixed amount but leave the impurity number unchanged. In the gauge theory these charges are understood to shift both the operator dimension and R-charge such that $\Delta = D - R$ remains fixed within the submultiplet. Acting with $L_{\rm sub}$ factors of these supercharges on a super-primary generates nine levels within each submultiplet labelled by $L_{\rm sub} = 0, \ldots, 8$. If the lowest energy level ($L_{\rm sub} = 0$) in the submultiplet is occupied by p degenerate super-primaries, the $L_{\rm sub}^{th}$ level will therefore contain $p\, C^8_{L_{\rm sub}}$ degenerate states, where C^m_n is the binomial coefficient. Furthermore, if the super-primary in a given submultiplet is a spacetime boson, the $L_{\rm sub}$ = even levels of the submultiplet will all be bosonic, and the $L_{\rm sub}$ = odd levels will be fermionic. The opposite is true if the bottom state is fermionic.

As an example, consider the one-loop, two-impurity supermultiplet structure studied in [10, 11]. The spectrum in this case contains only a single multiplet built on a scalar super-primary (labelled by 1_B, where the subscript denotes a bosonic level) with $O(1/J)$ one-loop energy shift $\Lambda^{(1)} = -6$. The $L_{\rm sub} = 1$ level therefore has eight degenerate states (8_F) with $\Lambda^{(1)} = -5$, the $L_{\rm sub} = 2$ level contains 28_B states with $\Lambda^{(1)} = -4$ and so on. We record the two-impurity supermultiplet structure in table 2 for comparison with higher-impurity spectra. The one-loop energies of the three protected $\mathfrak{sl}(2)$, $\mathfrak{su}(2)$ and $\mathfrak{su}(1|1)$ subsectors

$L_{\rm sub}$	0	1	2	3	4	5	6	7	8
	1_B	8_F	28_B	56_F	$\boxed{70_B}$	56_F	28_B	8_F	1_B
$\Lambda^{(1)}(L_{\rm sub})$	-6	-5	-4	-3	-2	-1	0	1	2
$\Lambda^{(2)}(L_{\rm sub})$	-4	-3	-2	-1	0	1	2	3	4

Table 2: Submultiplet breakup of the 256-dimensional two-impurity spectrum

studied here are degenerate in the two-impurity regime and lie in the boxed 70_B "centroid" level in table 2. We also record in table 2 the two-loop energy shifts $\Lambda^{(2)}$, which are offset from the one-loop values by two: $\Lambda^{(2)} = \Lambda^{(1)} + 2$.

In the gauge theory there are sixteen operators which increment the impurity number by one and shift the R-charge by certain amounts [26]. Four of these act on single-trace operators by rotating the $SO(6)$ scalars Z (carrying one unit of R-charge) into ϕ (which carry zero R-charge): they increase the operator impurity number by one and decrease the R-charge by one ($N \to N+1,\ R \to R-1$). Four operators rotate Z into $\mathcal{D}Z$, increasing N by one and leaving the R-charge fixed. The remaining eight operators are fermionic and send $N \to N+1,\ R \to R+1/2$. If one uses these operators to generate N-impurity super-primaries from those in the $(N-1)$-impurity spectrum, an immediate implication is that, within a given N-impurity spectrum of anomalous dimensions, all of the eigenvalues in the gauge theory will be related to each other by half-integer shifts in the R-charge. Certain energy levels will therefore be common to all of the submultiplets in the spectrum built on super-primary operators, and this special degeneracy can be used to deduce the overall structure of the extended supermultiplet. This degeneracy, however, only persists in the string theory to two-loop order in λ', and it is for this reason that we are forced to limit the general superstring spectral decomposition to two-loop order in the expansion. (It will be shown below, however, that a certain subset of submultiplets in the string theory can always be determined to *all* orders in λ'.)

Sending $J \to J + A$ on the string side (dual to an R-charge shift in the gauge theory) shifts $\Lambda^{(1)}$ and $\Lambda^{(2)}$ by $-2A$: starting from the two-impurity super-primary (1_B) with energy $\Lambda^{(1)} = -6$, the string versions of the sixteen impurity-increasing operators can be understood to generate four (degenerate) bosonic three-impurity super-primaries with $\Lambda^{(1)} = -8$, eight fermionic three-impurity super-primaries with $\Lambda^{(1)} = -7$ and four bosonic three-impurity super-primaries with $\Lambda^{(1)} = -6$. By acting with the eight charges $\mathcal{Q}_\alpha$ we then generate submultiplets based on each of these super-primaries whose levels are populated by $p\,C^8_{L_{\rm sub}}$ degenerate states, where p here is either four (for the two four-dimensional bosonic super-primary levels) or eight (for the eight-dimensional fermionic super-primary level). The submultiplets themselves can be labelled by a separate index K, in this case running over $K = 0, \ldots, 2$.

The complete three-impurity multiplet structure is recorded in table 3. Here there are a total of 11 levels in the extended supermultiplet, and we label these with the index L such that $L = L_{\rm sub} + K$. In table 3 the closed $\mathfrak{su}(2)$ sector lies in the boxed 280_B level in the $K = 0$ submultiplet with $\Lambda^{(1)} = -4$, the $\mathfrak{sl}(2)$ eigenvalue ($\Lambda^{(1)} = -2$) is in the boxed 280_B level of the $K = 2$ submultiplet and the $\mathfrak{su}(1|1)$ eigenvalue ($\Lambda^{(1)} = -3$) is in the 560_F level of the $K = 1$ submultiplet. For any impurity number these protected eigenvalues will always lie at the $L_{\rm sub} = 4$ level within their respective submultiplets. We also note that, in the K direction, the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ eigenvalues will correspond to eigenstates composed purely of S^5 or AdS_5 bosonic excitations, and will therefore fall into the "bottom" and "top" submultiplets, respectively (the $K = 0$ and $K = 2$ levels in the three-impurity case). Similarly, the $\mathfrak{su}(1|1)$ eigenvalue will correspond to eigenstates composed of either $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ or $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ excitations, and always lie in the "centroid" submultiplet in the K direction (the $K = 1$ level for three impurities). The energies shared by each of the submultiplets can be collected into degenerate levels of the complete supermultiplet. This total level degeneracy

$D(L)$ is recorded in the bottom row of table 3.

$K\backslash L$	0	1	2	3	4	5	6	7	8	9	10
0	4	32	112	224	$\boxed{280}$	224	112	32	4		
1		8	64	224	448	$\boxed{560}$	448	224	64	8	
2			4	32	112	224	$\boxed{280}$	224	112	32	4
$\Lambda^{(1)}(L)$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
$\Lambda^{(2)}(L)$	-6	-5	-4	-3	-2	-1	0	1	2	3	4
$D(L)$	4_B	40_F	180_B	480_F	840_B	1008_F	840_B	480_F	180_B	40_F	4_B

Table 3: Submultiplet breakup of the 4,096-dimensional three-impurity spectrum

It is easy to generalize this supermultiplet structure to arbitrary impurity number based on how the complete three-impurity spectrum is generated from the two-impurity supermultiplet above. For N impurities, the complete supermultiplet will have a total of 16^N states and $5+2N$ levels: the supermultiplet level index L therefore runs over $L = 0, \ldots, (4+2N)$. The entire supermultiplet breaks into $2N-3$ submultiplets, each of which have nine sublevels labelled by $L_{\rm sub} = 0, \ldots, 8$. The submultiplets themselves are labelled by the index K, which runs over $K = 0, \ldots, (2N-4)$. The one-loop energy shifts within the K^{th} submultiplet at level $L_{\rm sub}$ are thus given by

$$\Lambda^{(1)}_{\rm sub}(K, L_{\rm sub}, N) = K + L_{\rm sub} - 2(N+1) \ . \tag{4.2}$$

Equivalently, the L^{th} level of the entire supermultiplet has energy shift

$$\Lambda^{(1)}(L, N) = L - 2(N+1) \ . \tag{4.3}$$

The number of degenerate states at level $L_{\rm sub}$ within the K^{th} submultiplet is

$$D_{\rm sub}(K, L_{\rm sub}, N) = 4^{N-2} C_K^{2N-4} C^8_{L_{\rm sub}} \ , \tag{4.4}$$

so that the total dimension of the K^{th} submultiplet is $256 \times 4^{N-2} C_K^{2N-4}$. By summing the submultiplet degeneracies over a given supermultiplet level L, the total number of degenerate states at level L in the supermultiplet is given (in terms of Euler's Γ function) by

$$D(L, N) = \frac{4^{N-2}\Gamma(2N+5)}{\Gamma(2N+5-L)\Gamma(1+L)} \ . \tag{4.5}$$

The level is bosonic when L is even and fermionic when L is odd. As a verification of this formula, we can check that the total number of states in the N-impurity supermultiplet is indeed

$$\sum_{L=0}^{4+2N} \frac{4^{N-2}\Gamma(2N+5)}{\Gamma(2N+5-L)\Gamma(1+L)} = 16^N \ . \tag{4.6}$$

As noted above, the one-loop N-impurity $\mathfrak{su}(2)$ energy corresponds to eigenstates that are composed purely of symmetric-traceless $(\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})$ excitations: since each of these excitations increment the angular momentum J by one, the energy eigenvalue must therefore lie within a submultiplet built on super-primary states that exhibit the lowest possible energy in the extended supermultiplet. In other words, the $\mathfrak{su}(2)$ eigenvalue always lies at level $L_{\rm sub}=4$ of the $K=0$ submultiplet and, using the general formula in eqn. (4.2), we see that it exhibits the one-loop energy shift

$$\Lambda^{(1)}_{S^5}(N)=\Lambda^{(1)}_{\rm sub}(K=0,L_{\rm sub}=4,N)=-2(N-1)\ . \tag{4.7}$$

As a cross-check on this result, we note that this agrees with the one-loop limit of the general $\mathfrak{su}(2)$ eigenvalue formula (with unequal mode indices) in eqn. (3.11) above (with $N_B=N$):

$$\delta E_{S^5}(\{n_i\},N,J)=-\frac{1}{2J}\sum_{\substack{j,k=1\\ j\neq k}}^{N}(n_j^2+n_k^2)\,\lambda'+O(\lambda'^2)=-\frac{1}{J}\sum_{j=1}^{N}(N-1)\,n_j^2\,\lambda'+O(\lambda'^2)\ . \tag{4.8}$$

(Note the prefactor of 1/2 in the definition of $\Lambda^{(1)}$ in eqn. (4.1).) At this point we also see that $\Lambda^{(1)}$ indeed appears as a coefficient of $\sum n_j^2$, as given in eqn. (4.1).

The N-impurity $\mathfrak{sl}(2)$ eigenvalue, composed entirely of $(\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})$ excitations, must lie in the "top" $K=2N-4$ submultiplet at $L_{\rm sub}=4$. This gives the one-loop energy shift

$$\Lambda^{(1)}_{AdS}(N)=-2\ . \tag{4.9}$$

To check this we use the general $\mathfrak{sl}(2)$ formula for completely unequal mode indices in eqn. (3.24), and again expand to one-loop order in λ':

$$\delta E_{AdS}(\{n_i\},N,J)\ \ =\ \ =\frac{1}{J}\sum_{\substack{j,k=1\\ j\neq k}}^{N}n_j\,n_k\,\lambda'+O(\lambda'^2)\ . \tag{4.10}$$

With the level-matching condition $\sum_{j=1}^{N}n_j=0$ this becomes

$$\delta E_{AdS}(\{n_i\},N,J)=-\frac{1}{J}\sum_{j=1}^{N}n_j^2\lambda'+O(\lambda'^2)\ , \tag{4.11}$$

which agrees perfectly with the prediction in eqn. (4.9) (and again confirms that $\Lambda^{(1)}$ here is a coefficient of $\sum n_j^2$).

Finally, the $\mathfrak{su}(1|1)$ one-loop eigenvalue, composed of either $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ or $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ spinors, lies in the $K=N-2$ submultiplet at $L_{\rm sub}=4$, exhibiting the one-loop energy shift

$$\Lambda^{(1)}_{\mathfrak{su}(1|1)}(N)=-N\ . \tag{4.12}$$

Using eqn. (3.30) we see that

$$\begin{aligned}\delta E_{su(1|1)}(\{n_i\},N,J) &= -\frac{1}{4J}\sum_{\substack{j,k=1\\ j\neq k}}^{N}(n_j-n_k)^2\lambda' + O(\lambda'^2) \\ &= -\frac{1}{2J}\sum_{j=1}^{N} N\, n_j^2\,\lambda' \ , \end{aligned} \tag{4.13}$$

where we have again invoked the level-matching condition to derive the last line. This of course agrees with eqn. (4.12). For reference we present in table 4 the complete 65,536-dimensional four-impurity spectrum of one- and two-loop energies. The $\mathfrak{su}(2)$ eigenvalue in this case lies in the boxed 1120_B level with $\Lambda^{(1)} = -6$, the $\mathfrak{su}(1|1)$ eigenvalue is in the 6720_B level with $\Lambda^{(1)} = -4$, and the $\mathfrak{sl}(2)$ energy lies in the 1120_B level with $\Lambda^{(1)} = -2$.

$K \setminus L$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	16	128	448	896	1120	896	448	128	16				
1		64	512	1792	3584	4480	3584	1792	512	64			
2			96	768	2688	5376	6720	5376	2688	768	96		
3				64	512	1792	3584	4480	3584	1792	512	64	
4					16	128	448	896	1120	896	448	128	16
$\Lambda^{(1)}(L)$	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
$\Lambda^{(2)}(L)$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4
$D(L)$	16_B	192_F	1056_B	3520_F	7920_B	12672_F	14784_B	12672_F	7920_B	3520_F	1056_B	192_F	16_B

Table 4: Submultiplet breakup of the 65,536-dimensional four-impurity spectrum

Comparing the $\Lambda^{(2)}$ and $\Lambda^{(1)}$ spectra in tables 2 and 3 (which are determined directly from the string Hamiltonian), we see that the spectrum of $\Lambda^{(2)}$ is identical to $\Lambda^{(1)}$ up to an overall shift. The two-loop analogue of the general N-impurity energy shift of eqn. (4.2) is therefore

$$\Lambda^{(2)}_{\rm sub}(K, L_{\rm sub}, N) = K + L_{\rm sub} - 2N \ . \tag{4.14}$$

Equivalently, we have $\Lambda^{(2)}(L, N) = L - 2N$ for the entire supermultiplet shift in terms of L.

Similar to the one-loop case, we can test this two-loop formula using the N-impurity results derived above in the three protected sectors. According to eqn. (4.14), the $\mathfrak{su}(2)$ eigenvalue in the $K = 0$ submultiplet at level $L_{\rm sub} = 4$ has the following two-loop energy shift:

$$\Lambda^{(2)}_{S^5}(N) = 4 - 2N \ . \tag{4.15}$$

Isolating the two-loop energy eigenvalue $\delta E^{(2)}_{S^5}$ from the N-impurity $\mathfrak{su}(2)$ equation (3.11), we have

$$\begin{aligned} \delta E^{(2)}_{S^5}(\{n_i\}, N, J) &= \frac{1}{4J} \sum_{\substack{j,k=1 \\ j\neq k}}^{N} (n_j^4 + n_j^3 n_k + n_j n_k^3 + n_k^4) \lambda'^2 \\ &= -\frac{1}{4J} \sum_{j=1}^{N} (n_j^4)(4 - 2N) \lambda'^2 \ , \end{aligned} \tag{4.16}$$

which matches our prediction. The $\mathfrak{sl}(2)$ eigenvalue in the $K = 2N - 4$ submultiplet is predicted to vanish

$$\Lambda^{(2)}_{AdS}(N) = 0 \ , \tag{4.17}$$

which agrees with the two-loop expansion term in eqn. (3.24):

$$\delta E^{(2)}_{AdS}(\{n_i\}, J) = -\frac{1}{4J} \sum_{\substack{j,k=1 \\ j\neq k}}^{N} \left[n_j n_k (n_j + n_k)^2 \right] \lambda'^2 = 0 \ . \tag{4.18}$$

Finally, the $\mathfrak{su}(1|1)$ pure-fermion sector in the $K = N - 2$ submultiplet at $L_{\rm sub} = 4$ should have an energy shift of

$$\Lambda^{(2)}_{\mathfrak{su}(1|1)}(N) = 2 - N \ , \tag{4.19}$$

which agrees with the $\mathfrak{su}(1|1)$ formula given in eqn. (3.30):

$$\begin{aligned} \delta E^{(2)}_{\mathfrak{su}(1|1)}(\{n_i\}, N, J) &= \frac{1}{8J} \sum_{\substack{j,k=1 \\ j\neq k}}^{N} (n_j^2 - n_k^2)^2 \lambda'^2 \\ &= -\frac{1}{4J} \sum_{j=1}^{N} (n_j^4)(2 - N) \lambda'^2 \ . \end{aligned} \tag{4.20}$$

As described in [12], it should also be noted that since we know the $\mathfrak{su}(2)$, $\mathfrak{sl}(2)$ and $\mathfrak{su}(1|1)$ eigenvalues to all orders in λ', we can easily determine complete all-loop energy formulas for the three submultiplets to which these eigenvalues belong. It was previously noted that the eight supercharges ($\mathcal{Q}_\alpha$) that act as raising operators within each submultiplet are known in the gauge theory to shift both the dimension and R-charge by $1/2$ such that $\Delta = D - R$ is kept fixed. Because all states within a given submultiplet share the same Δ, the string energy shift at any level $L_{\rm sub}$ can therefore be obtained from that at level $L'_{\rm sub}$ by replacing

$$J \to J - L_{\rm sub}/2 + L'_{\rm sub}/2$$

in the energy eigenvalue evaluated at sub-level $L'_{\rm sub}$. Since we are expanding to $O(J^{-1})$, however, this replacement can only affect the eigenvalues δE via the $O(J^0)$ BMN term in the pp-wave limit. For the protected eigenvalues determined above at $L_{\rm sub} = 4$, we therefore find the all-loop energy shift for the entire submultiplet by including the appropriate $O(J^{-1})$ contribution from the BMN formula

$$E_{\rm BMN} = \sum_{j=1}^{N} \sqrt{1 + \frac{n_j^2 \lambda}{(J + 2 - L_{\rm sub}/2)^2}} \ . \tag{4.21}$$

Explicitly, the complete level spectra of the $K = 0$, $K = N-2$ and $K = 2N-4$ submultiplets are given, to all orders in λ', by

$$\delta E(\{n_j\}, L_{\rm sub}, N, J) = \frac{\lambda'}{2J} \sum_{j=1}^{N} \frac{n_j^2 (L_{\rm sub} - 4)}{\sqrt{1 + n_j^2 \lambda'}} + \delta E_{L_{\rm sub}=4}(\{n_j\}, J) \ , \tag{4.22}$$

where $\delta E_{L_{\rm sub}=4}$ is the $L_{\rm sub} = 4$ energy shift in the submultiplet of interest. Since the level degeneracy among submultiplets is generally broken beyond two-loop order, it is difficult to obtain similar expressions for submultiplets not containing the $\mathfrak{su}(2)$, $\mathfrak{sl}(2)$ and $\mathfrak{su}(1|1)$ protected eigenvalues. This can possibly be addressed by relying directly on the commutator algebra of various impurity-increasing operators in the string theory, and we will return to this problem in a future study.

5 Discussion

We have directly computed the near-pp-wave eigenvalues of N-impurity bosonic string states with arbitrary mode-number assignment lying in the protected symmetric-traceless irreps of the AdS_5 ($\mathfrak{sl}(2)$) and S^5 ($\mathfrak{su}(2)$) subspaces. Based on the observation that the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ eigenvalues are simple generalizations of the three-impurity results obtained in [12], we have also presented a conjecture for the N-impurity eigenvalues of symmetrized-fermion states in the $\mathfrak{su}(1|1)$ sector. This conjecture meets several basic expectations and we believe that it is correct. (It would be satisfying, however, to derive the $\mathfrak{su}(1|1)$ eigenvalue formula directly

from the fermionic sector of the string theory.) We have also found that the $\mathfrak{su}(2)$ eigenvalues perfectly match, to all orders in λ', the corresponding eigenvalue predictions given by the string Bethe ansatz of [19]. Along these lines, it would be very interesting to have long-range Bethe ansätze analogous to the string [19] and gauge theory [7, 8] $\mathfrak{su}(2)$ equations for either the protected $\mathfrak{sl}(2)$ and $\mathfrak{su}(2|3)$ sectors or for the entire $\mathfrak{psu}(2,2|4)$ algebra of the theory.

The supermultiplet decomposition given in section 4 is based on the breakup of the energy spectrum observed between the two- and three-impurity regime, and is precisely what is expected from the gauge theory based on how sixteen particular charges are known to act on operators which are dual to the string states of interest [12, 26]. Assuming that this mechanism is not specific to the three-impurity case, we were able to generalize the decomposition of the N-impurity (unequal mode index) supermultiplet to two-loop order in λ'. By knowing where the eigenvalues of the $\mathfrak{su}(2)$, $\mathfrak{sl}(2)$ and $\mathfrak{su}(1|1)$ sectors are supposed to appear in this decomposition, we were able to provide a stringent cross-check of our results, and we have found perfect agreement. Given the many implicit assumptions in this procedure, however, it would be instructive to perform a direct diagonalization of the four-impurity Hamiltonian to test our predictions. While such a test is likely to be computationally intensive, the problem could be simplified to some extent by restricting to the pure-boson H_{BB} sector at one loop in λ'. We of course expect complete agreement with the results presented here.

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